

LXXIV. Spherical Trigonometry *reduced to Plane*, by Francis Blake, *Esq. F. R. S.*

Read May 7, 1752. IT is observable, that the analogies of spherical trigonometry, exclusive of the terms co-fine and co-tangent, are applicable to plane, by only changing the expression, fine or tangent of side, into the single word, side\*: so that the business of plane trigonometry, like a corollary to the other, is thence to be inferr'd. And the reason of this is obvious; for analogies raised not only from the consideration of a triangular figure, but the curvature also, are of consequence more general; and tho' the latter should be held evanescent by a diminution of the surface, yet what depends upon the triangle, will nevertheless remain. These things may have been observed, I say; but upon revising the subject, it further occur'd to me, and I take it to be new, that from the axioms of only plane trigonometry, and almost independent of solids, and the doctrine of the sphere, the spherical cases are likewise to be solved.

Suppose, first, that the three sides of a spherical triangle,  $abd$  (Fig. 1.) are given to find an angle,  $a$ ; which case will lay open the method, and lead on to the other cases, in a way, that to me appears the most natural. It is allow'd, that the tangents,  $ae$ ,  $af$ , of the sides,  $ad$ ,  $ab$ , including an angle,  $a$ , make a plane angle equal to it; and it is evident, that the other side,  $db$ , determines the angle made by the secants  $ce$ ,  $cf$ , at  $c$  the centre of the sphere; whence the distance,  $ef$ , betwixt the tops of those secants, is

K k k given

\* See M. De la Caille's remark at the end of the spherical trigonometry prefix'd to his Elements of Astronomy.

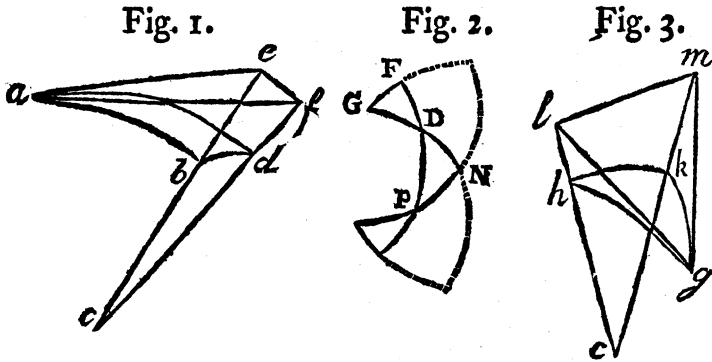
given by case the fifth of oblique plane triangles (see *Heynes's Trigonom.*) which, with the aforefaid tangents, reduces it to case the 6th of oblique plane triangles also \* : and thus this 11th case of oblique triangles, so intricate hitherto, becomes perfectly easy. The 12th case is reducible to the 11th, and the rest, whether right-angled, or oblique, we are authoris'd to look upon as reducible to right-angled triangles, whose sides are not quadrants, but either greater or less than such. Conceive therefore, now, in a right-angled spherical triangle,  $gkb$  (Fig. 2.) that the tangent,  $gm$ , and secant,  $em$ , of either leg,  $gk$ , is already drawn; and in the point,  $m$ , of their union, draw a perpendicular,  $ml$ , to  $em$ , the secant, directly above the other leg, *viz.* a perpendicular to the plane of the secant and tangent, that it may be perpendicular to both (*Eucl.* 4, 11); for then will the tangent,  $gl$ , of the hypotenuse,  $gb$ , drawn from the same point, which that of the leg was, constantly terminate in the perpendicular line, that the radius and tangent may make a right-angle (*Eucl.* 18, 3). Whence these tangents,  $gm$ ,  $gl$ , and the perpendicular line,  $ml$ , together with the secants,  $cm$ ,  $cl$ , will evidently form two right-angled plane triangles,  $gml$ ,  $cm l$ ; and to one or other of these the spherical cases are easily transferr'd. Thus, if in the spherical triangle,  $gkb$ , the hypotenuse,  $gb$ , base,  $gk$ , and angle,  $g$ , at the base, be the parts given and required, when any two are given, the  
third

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\* The angle to be found in this case must always be that formed by the two tangents.

third may be determined by means of a plane triangle; and at a single operation. We have, for instance, in the right-angled plane triangle,  $gml$ , formed as above, the base,  $gm$ , and hypotenuse,  $gl$ , to find, by case the 5th of right-angled plane triangles, the angle included, which is the same as on the sphere. And then if the base,  $gk$ , the angle,  $g$ , at the base, and perpendicular,  $kb$ , be the spherical parts given and required; or if the angles,  $g$  and  $b$ , and the hypotenuse,  $gb$ , be the parts given and required, we have only that former proportion of the hypotenuse and base, and angle at the base, in the triangles,  $PND$ ,  $DFG$ , obtained by the complements, to transfer to the plane. But secondly, suppose the spherical proportion is of the three sides, any two being given, the third may be also found at a single operation, in the second right-angled plane triangle,  $cm l$ , form'd as above. We have, for instance, the hypotenuse and base,  $cl$ ,  $cm$ , viz. the secant of the spherical hypotenuse and base  $gb$ ,  $gk$ , to find, by the 5th of right-angled plane triangles, the angle,  $c$ , at the center, which is the measure of  $kb$ , the side that was sought. And then again, if the hypotenuse, one leg, and the opposite angle be the spherical parts given and required; or if the two angles and a leg be the parts given and required, we have only the former proportion of the three sides in the triangles,  $PND$ ,  $DFG$ , obtained by the complements, to transfer to the plane. Whence, the six proportions of right-angled spherical triangles being comprehended in this method, it is fully demonstrated, that all the cases of these triangles are so to be resolved.

The same might be deduced without the method of complements, but neither in so short nor satisfactory a way, and it shall therefore be omitted. I have communicated this upon account of its perspicuity, and supposing, that in an age so greatly advanced in mathematical learning, the least hint of what is new would not be unacceptable.



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